Int. J. Solids Structures, 1974, Vol. 10, pp. 1103–1123. Pergamon Press. Printed in Great Britain.

BOUNDARY LAYERS IN HIGHLY ANISOTROPIC PLANE ELASTICITY

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(*Received* 3 *December* 1973; *revised* 25 *February* 1974)

Abstract—The boundary layer method proposed by Everstine and Pipkin for the analysis of highly anisotropic materials, such as fibre-reinforced materials, in elastic plane strain is developed and extended also to include plane stress. It is applied to problems of point forces acting on half-planes, and to two crack problems. The boundary layer solutions are compared with known exact solutions in anisotropic elasticity, and it is found that the boundary layer theory gives good results for elastic constants typical of a carbon fibre reinforced resin.

1. INTRODUCTION

Materials such as resins reinforced by strong aligned fibres exhibit highly anisotropic elastic behaviour in the sense that their elastic moduli for extension in the fibre direction are frequently of the order of 50 or more times greater than their elastic moduli in transverse extension or in shear. An idealization of this property is to assume that the material is inextensible in the fibre direction, and a simple theory has been constructed on this basis[1–4]. A feature of solutions using this theory, which has been illustrated in many examples, is the occurrence of singular fibres or sheets of fibres, which support finite forces, and consequently infinite stress, and transmit force for large distances without attenuation. By considering some simple but illuminating examples Everstine and Pipkin[5] showed that in plane elasticity for a highly anisotropic but not inextensible material these singular fibres become regions of stress concentration which attenuates slowly with distance along the fibre, and provided estimates for the stress in and dimensions of these layers. Subsequently Everstine and Pipkin[6] formulated a boundary layer analysis for highly anisotropic materials in plane strain, and applied it to the deflection of a cantilever beam under end load.

The purpose of this paper is to further develop and apply this boundary layer theory proposed by Everstine and Pipkin[6] and compare its predictions with some exact solutions in anisotropic elasticity.

The essential theory of plane anisotropic elasticity is summarized in Section 2. Both plane strain and plane stress are considered. The corresponding inextensible theory is outlined in Section 3. Unlike some previous work, it is not necessarily assumed that the material is incompressible as well as inextensible in the fibre direction. The boundary layer equations are formulated in Section 4. The formulation differs a little from that of[6], but the end results are essentially the same although the present theory includes plane stress as well as plane strain problems. The problem reduces to the solution of Laplace's equation for the two displacement components, in appropriately scaled coordinates.

The remainder of the paper is devoted to applications. Basic point force problems for the half-plane are solved in Section 5, and compared with known exact solutions. The problem of a crack parallel to the fibres is shear is solved in Section 6. In Section 7 the boundary layer solution is obtained for a crack normal to the fibre direction opened by internal pressure. This solution is compared with the exact anisotropic elastic solution, and very satisfactory agreement is obtained for elastic constants typical of a carbon fibre reinforced resin.

2. PLANE PROBLEMS FOR TRANSVERSELY ISOTROPIC MATERIALS

The constitutive equation for a transversely isotropic linear elastic material whose preferred direction is that of a unit vector **a** is

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + e_{kk}a_i a_j) + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta a_k a_m e_{km} a_i a_j. \quad (2.1)$$

Here t_{ij} are components of stress, e_{ij} components of infinitesimal strian, a_i the components of **a**, all referred to rectangular cartesian coordinates x_i. The vector **a** may be a function of position. Indices take the values 1, 2, 3 and summation convention is employed. The coefficients λ, μ_T, μ_L, α, β are elastic constants with the dimensions of stress.

If **a** is chosen so that its components are (1, 0, 0), so that the preferred direction is everywhere that of the x_1 axis, (2.1) become

$$\begin{aligned} t_{11} &= (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)e_{11} + (\lambda + \alpha)e_{22} + (\lambda + \alpha)e_{33}, \\ t_{22} &= (\lambda + \alpha)e_{11} + (\lambda + 2\mu_T)e_{22} + \lambda e_{33}, \\ t_{33} &= (\lambda + \alpha)e_{11} + \lambda e_{22} + (\lambda + 2\mu_T)e_{33}, \\ t_{23} &= 2\mu_T e_{23},\ t_{13} = 2\mu_L e_{13},\ t_{12} = 2\mu_L e_{12}. \end{aligned} \quad (2.2)$$

From these it is seen that μ_T can be identified as the shear modulus in transverse shear across the preferred direction, and μ_L as the shear modulus in longitudinal shear in the preferred direction. The other constants λ, α, β can also be related to more familiar quantities, for from (2.2) it can be shown that

$$\begin{aligned} E &= \frac{(\lambda + \mu_T)\beta' - (\lambda + \alpha)^2}{\lambda + \mu_T}, \qquad E' = \frac{4\mu_T[(\lambda + \mu_T)\beta' - (\lambda + \alpha)^2]}{(\lambda + 2\mu_T)\beta' - (\lambda + \alpha)^2}, \\ \nu &= \frac{\lambda + \alpha}{2(\lambda + \mu_T)}, \qquad \nu' = \frac{\lambda\beta' - (\lambda + \alpha)^2}{(\lambda + 2\mu_T)\beta' - (\lambda + \alpha)^2}, \end{aligned} \quad (2.3)$$

where

$$\beta' = \lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta. \quad (2.4)$$

Here E is the extensional modulus for uniaxial tension in the direction of **a**, ν the corresponding Poisson's ratio, E' the extensional modulus for uniaxial tension in a direction normal to α, and ν' is the ratio $-e_{33}/e_{22}$ for uniaxial tension in the x_2 direction. The constants E, E', ν, ν' and $G = \mu_L$ are those used by Everstine and Pipkin[5,6]; they differ from those used by Lekhnitskii[7] by interchange of primed and unprimed quantities.

In plane strain in the (x_1, x_2) plane, $e_{13} = e_{23} = e_{33} = 0$, and (2.2) give

$$t_{11} = \beta' e_{11} + (\lambda + \alpha)e_{22}, \qquad t_{22} = (\lambda + \alpha)e_{11} + (\lambda + 2\mu_T)e_{22}, \qquad t_{12} = 2\mu_L e_{12}. \quad (2.5)$$

Similarly in plane stress, $t_{13} = t_{23} = t_{33} = 0$,

$$
\begin{aligned}
t_{11} &= \frac{(\lambda + 2\mu_T)\beta' - (\lambda + \alpha)^2}{\lambda + 2\mu_T} e_{11} + \frac{2\mu_T(\lambda + \alpha)}{\lambda + 2\mu_T} e_{22}, \\
t_{22} &= \frac{2\mu_T(\lambda + \alpha)}{\lambda + 2\mu_T} e_{11} + \frac{4\mu_T(\lambda + \mu_T)}{\lambda + 2\mu_T} e_{22}, \\
t_{12} &= 2\mu_L e_{12}.
\end{aligned} \tag{2.6}
$$

Thus in either case the equations are of the form

$$t_{11} = Le_{11} + Me_{22}, \qquad t_{22} = Me_{11} + Ne_{22}, \qquad t_{12} = 2\mu_L e_{12}, \tag{2.7}$$

where L, M, N can be related to the elastic constants by comparison with (2.5) or (2.6) as appropriate.

In considering plane problems the coordinates x_1, x_2 will be replaced by x, y respectively. The stress components in the (x, y) plane are then denoted by t_{xx}, t_{yy}, t_{xy}, and the corresponding strain components by e_{xx}, e_{yy}, e_{xy}. Components of displacement in the x and y directions will be denoted u, v, so that

$$e_{xx} = \partial u/\partial x, \qquad e_{yy} = \partial v/\partial y, \qquad e_{xy} = \tfrac{1}{2}(\partial u/\partial y + \partial v/\partial x). \tag{2.8}$$

In this notation (2.7) become

$$t_{xx} = Le_{xx} + Me_{yy}, \qquad t_{yy} = Me_{xx} + Ne_{yy}, \qquad t_{xy} = 2\mu_L e_{xy}. \tag{2.9}$$

The trajectories of the vector **a** will, for convenience, be called fibres, and its direction the fibre direction. The preferred direction is everywhere tangential to the fibres. In the present case with $\mathbf{a} = (1, 0, 0)$, the fibres are all straight lines parallel to the x-axis. This terminology is suggested by the possible application of the theory to materials reinforced with a family of aligned fibres, but the theory of this section is applicable to any transversely isotropic linear elastic material.

3. THE INEXTENSIBLE THEORY

The materials under consideration are ones which have (in an appropriate sense defined below) a large modulus for extension in the fibre direction. A simple theory can be constructed[3,4] by idealizing this property by making the assumption that the material is inextensible in the fibre direction, so that

$$a_i a_j e_{ij} = 0. \tag{3.1}$$

The normal stress component corresponding to the fibre direction is then an arbitrary tension which is a reaction to the constraint of inextensibility, and the constitutive equation takes the form

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + T a_i a_j, \tag{3.2}$$

where T represents the arbitrary fibre tension. Comparing (2.1) and (3.2), it is evident that this inextensible theory requires the limit $\beta \to \infty$.

If in addition the material is incompressible, then also

$$e_{kk} = 0, \tag{3.3}$$

the hydrostatic pressure becomes an arbitrary reaction to this constraint, and the constitutive equation is

$$t_{ij} = -p\delta_{ij} + 2\mu_T e_{ij} + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + Ta_i a_j, \tag{3.4}$$

where p represents the hydrostatic pressure. Evidently (3.4) arises from (3.2) in the limit $\lambda \to \infty$. The theory then becomes that of the ideal fibre-reinforced solid described by Pipkin and Rogers[1] and Spencer[2]. However, although incompressibility is often a valid approximation to make when dealing with finite deformations of solids, it is less often so in small deformation theories, and so the less restrictive assumption (3.2) is adopted here. Nevertheless, many of the results will remain true in the limit $\lambda \to \infty$ of an incompressible material, and it is also worth noting that in many problems in linear elasticity values of the quantities of interest are not sensitive to the compressibility of the material. The simpler incompressible theory is therefore of considerable value in certain cases.

For plane strain, and $\mathbf{a} = (1, 0, 0)$, (3.2) give

$$t_{xx} = T, \qquad t_{yy} = (\lambda + 2\mu_T)e_{yy}, \qquad t_{xy} = 2\mu_L e_{xy}, \tag{3.5}$$

and similarly, for plane stress, (3.2) give

$$t_{xx} = T, \qquad t_{yy} = \frac{4\mu_T(\lambda + \mu_T)}{\lambda + 2\mu_T} e_{yy}, \qquad t_{xy} = 2\mu_L e_{xy}. \tag{3.6}$$

In both (3.5) and (3.6) additional terms have, without loss of generality, been absorbed into the arbitrary tension T. If the material is incompressible then $\lambda \to \infty$. Then in plane strain $e_{yy} = 0$, and (3.5) become

$$t_{xx} = T_1, \qquad t_{yy} = T_2, \qquad t_{xy} = 2\mu_L e_{xy},$$

and T_1 and T_2 are both arbitrary. These are also the appropriate equations for plane strain of a material reinforced by two families of inextensible fibres[2,8]. In the case of plane stress, however, incompressibility does not imply $e_{yy} = 0$, and in the limit $\lambda \to \infty$ (3.6) become

$$t_{xx} = T, \qquad t_{yy} = 4\mu_T e_{yy}, \qquad t_{xy} = 2\mu_L e_{xy}. \tag{3.7}$$

Thus, as noted by England, Ferrier and Thomas[3], for either compressible plane strain, or compressible or incompressible plane stress, the equations take the form

$$t_{xx} = T, \qquad t_{yy} = Ne_{yy}, \qquad t_{xy} = 2\mu_L e_{xy}, \tag{3.8}$$

where

$$N = \begin{cases} \lambda + 2\mu_T & \text{(compressible plane strain)}, \\ \dfrac{4\mu_T(\lambda + \mu_T)}{\lambda + 2\mu_T} & \text{(compressible plane stress)}, \\ 4\mu_T & \text{(incompressible plane stress)}. \end{cases} \tag{3.9}$$

Since $e_{xx} = \partial u/\partial x = 0$, there follows

$$u = f(y). \tag{3.10}$$

The equilibrium equations

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} = 0, \qquad \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} = 0, \tag{3.11}$$

give, with (3.8) and (3.10)

$$\frac{\partial T}{\partial x} + \mu_L \left[u''(y) + \frac{\partial^2 v}{\partial x \, \partial y} \right] = 0, \tag{3.12}$$

$$\mu_L \frac{\partial^2 v}{\partial x^2} + N \frac{\partial^2 v}{\partial y^2} = 0. \tag{3.13}$$

For convenience introduce a constant

$$c^2 = \mu_L / N, \tag{3.14}$$

and then v satisfies

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{c^2} \frac{\partial^2 v}{\partial y^2} = 0. \tag{3.15}$$

A number of solutions of these equations, as well as a more detailed derivation of them, are given in[3]. In the incompressible limit for plane strain, but not for plane stress, $c \to 0$. For $c \neq 0$, it is convenient to introduce a new variable

$$\xi = x/c,$$

in terms of which (3.15) becomes

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial y^2} = 0, \tag{3.16}$$

so that v is a harmonic function of ξ and y.

4. BOUNDARY LAYER EQUATIONS

A feature of solutions of problems using the inextensible theory of the preceding section is that they frequently predict the existence of singular sheets of fibres, which carry infinite stress but finite force. This was first pointed out by Pipkin and Rogers[1] and has since been demonstrated in numerous examples. These singular fibres may occur either adjacent to the surface of a body or in its interior. In the case of incompressible plane strain it is also possible for the normal curves, that is the orthogonal trajectories of the fibres, to be singular. Everstine and Pipkin[5] demonstrated convincingly by considering some simple examples that in reality these singular sheets of fibres represent narrow bands of intense stress concentration. If l is a characteristic length of a problem, these bands have width of order $(\mu_L/E)^{1/2} l$, and along them the stress decays in a length of order $(\mu_L/E)^{-1/2} l$. The inextensible theory corresponds to the limit $\mu_L/E \to 0$. Thus for $\mu_L/E \ll 1$ the singular fibres represent boundary layers across which certain stress components vary rapidly, and Everstine and Pipkin[5] pointed out that the equations are of a suitable form for the application of a boundary layer or singular perturbation analysis. Subsequently[6] they developed such an analysis and applied it to the problem of the deflection of a cantilever beam under end load. The purpose of this paper is to give some further developments and applications of this boundary layer theory.

Everstine and Pipkin[5] introduced the Airy stress function χ which for anisotropic plane elasticity satisfies the generalized biharmonic equation

$$C \frac{\partial^4 \chi}{\partial x^4} + B \frac{\partial^4 \chi}{\partial x^2 \, \partial y^2} + A \frac{\partial^4 \chi}{\partial^4 y} = 0. \tag{4.1}$$

For the isotropic case $A = \frac{1}{2}B = C = 1$. The inextensible theory arises in the limit $A/B \to 0$. We proceed somewhat differently directly from equations (2.9); the end result is effectively the same. Substituting the stress components (2.9) in the equilibrium equations (3.11) gives

$$
\begin{aligned}
L\frac{\partial^2 u}{\partial x^2} + (M + \mu_L)\frac{\partial^2 v}{\partial x\,\partial y} + \mu_L\frac{\partial^2 u}{\partial y^2} &= 0,\\
\mu_L\frac{\partial^2 v}{\partial x^2} + (M + \mu_L)\frac{\partial^2 u}{\partial x\,\partial y} + N\frac{\partial^2 v}{\partial y^2} &= 0.
\end{aligned}
\tag{4.2}
$$

Now introduce the notation

$$
\frac{\mu_L}{L} = \epsilon^2, \qquad \frac{\mu_L}{M} = d^2, \qquad \frac{\mu_L}{N} = c^2,\dagger \tag{4.3}
$$

[the last of these is a repetition of (3.14)]. Then (2.9) may be written

$$
\frac{t_{xx}}{\mu_L} = \frac{1}{\epsilon^2}e_{xx} + \frac{1}{d^2}e_{yy}, \qquad \frac{t_{yy}}{\mu_L} = \frac{1}{d^2}e_{xx} + \frac{1}{c^2}e_{yy}, \qquad \frac{t_{xy}}{\mu_L} = 2e_{xy} \tag{4.4}
$$

and (4.2) become

$$
\begin{aligned}
\frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial x^2} + \left(1 + \frac{1}{d^2}\right)\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} &= 0,\\
\frac{\partial^2 v}{\partial x^2} + \left(1 + \frac{1}{d^2}\right)\frac{\partial^2 u}{\partial x\,\partial y} + \frac{1}{c^2}\frac{\partial^2 v}{\partial y^2} &= 0.
\end{aligned}
\tag{4.5}
$$

In the limit of an inextensible compressible material $\beta \to \infty$ while λ, μ_T, μ_L and α remain fixed. It follows from (2.4), (2.5) and (2.6) that in this limit, for either plane strain or plane stress, $L \to \infty$ with μ_L, M and N held constant. It further follows from (4.3) that $\epsilon \to 0$, with d and c constant. It is supposed for the present that ϵ is a small but finite parameter, corresponding to a material which is "almost inextensible". To illustrate how these assumptions relate to certain real materials, we refer to measurements by Markham[11] of the elastic constants of a carbon fibre-expoxy resin composite. Markham obtains the following values for the components c_{ij} of the stiffness matrix (with the 3 axis corresponding to the fibre direction)

$$
\begin{aligned}
&c_{11} = 10{\cdot}57 \times 10^9\,\mathrm{Nm}^{-2}, \quad c_{33} = 241{\cdot}71 \times 10^9\,\mathrm{Nm}^{-2}, \quad c_{12} = 5{\cdot}64 \times 10^9\,\mathrm{Nm}^{-2},\\
&c_{13} = 4{\cdot}37 \times 10^9\,\mathrm{Nm}^{-2}, \quad c_{44} = \mu_L = 5{\cdot}66 \times 10^9\,\mathrm{Nm}^{-2}, \quad c_{66} = \mu_T = 2{\cdot}46 \times 10^9\,\mathrm{Nm}^{-2}.
\end{aligned}
$$

The corresponding values of E, E', ν (also quoted by Bishop[9]) and ν' are

$$
E = 239 \times 10^9\,\mathrm{Nm}^{-2}, \qquad E' = 7{\cdot}54 \times 10^9\,\mathrm{Nm}^{-2}, \qquad \nu = 0{\cdot}27, \qquad \nu' = 0{\cdot}53.
$$

These give, for plane strain

$$
\begin{aligned}
L = c_{33} = 241{\cdot}71 \times 10^9\,\mathrm{Nm}^{-2}, \qquad M = c_{13} = 4{\cdot}37 \times 10^9\,\mathrm{Nm}^{-2},\\
N = c_{11} = 10{\cdot}57 \times 10^9\,\mathrm{Nm}^{-2},
\end{aligned}
$$

and for plane stress

† The constants ϵ and c differ slightly from the similarly denoted quantities in[6].

$$L = (c_{11}c_{33} - c_{13}^2)/c_{11} = 239{\cdot}9 \times 10^9 \mathrm{Nm}^{-2},$$
$$M = c_{13}(c_{11} - c_{12})/c_{12} = 2{\cdot}04 \times 10^9 \mathrm{Nm}^{-2},$$
$$N = (c_{11}^2 - c_{12}^2)/c_{11} = 7{\cdot}56 \times 10^9 \mathrm{Nm}^{-2}.$$

Then from (4.3) the values of the parameters ϵ, c and d are

Plane strain.	$\epsilon^2 = 0{\cdot}023$,	$c^2 = 0{\cdot}54$,	$d^2 = 1{\cdot}29$.
Plane stress.	$\epsilon^2 = 0{\cdot}024$,	$c^2 = 0{\cdot}75$,	$d^2 = 2{\cdot}76$.

Thus for this material it is not unreasonable to seek approximations based on the assumption that ϵ is small, while c and d are of order one, although it will be shown below that less restrictive conditions on c and d will suffice.

In plane strain (but not in plane stress) if the material is incompressible as well as inextensible, then $c = 0$. Correspondingly for an "almost incompressible" material in plane strain c, as well as ϵ, is a small parameter. For the present we place no restrictions on the magnitude of c.

In the neighbourhood of a fibre which according to the inextensible theory is a singular fibre, quantities may vary rapidly in the y-direction, which is normal to the fibres. Following usual boundary-layer theory procedures, to accommodate such variations we "stretch" the y-coordinate, the appropriate scaling factor being ϵ. Hence we substitute

$$y = \epsilon\eta. \tag{4.6}$$

Equations (4.4) and (4.5) then become

$$\frac{t_{xx}}{\mu_L} = \frac{1}{\epsilon^2}\frac{\partial u}{\partial x} + \frac{1}{\epsilon d^2}\frac{\partial v}{\partial \eta},$$
$$\frac{t_{yy}}{\mu_L} = \frac{1}{d^2}\frac{\partial u}{\partial x} + \frac{1}{\epsilon c^2}\frac{\partial v}{\partial \eta}, \tag{4.7}$$
$$\frac{t_{xy}}{\mu_L} = \frac{1}{\epsilon}\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial x},$$

$$\frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon}\left(1 + \frac{1}{d^2}\right)\frac{\partial^2 v}{\partial x\,\partial \eta} + \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial \eta^2} = 0, \tag{4.8}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{\epsilon}\left(1 + \frac{1}{d^2}\right)\frac{\partial^2 u}{\partial x\,\partial \eta} + \frac{1}{\epsilon^2 c^2}\frac{\partial^2 v}{\partial \eta^2} = 0. \tag{4.9}$$

Everstine and Pipkin[6] show that in the boundary layer u/v is of order ϵ, but $(\partial u/\partial x)/(\partial v/\partial \eta)$ is of order $1/\epsilon$. Hence the terms of lowest order in ϵ in (4.8) and (4.9) give

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2} = 0, \qquad \frac{\partial^2 v}{\partial \eta^2} = 0 \tag{4.10}$$

provided that

$$d = O(\epsilon^p), \qquad c = O(\epsilon^q), \qquad \text{where } p < 1, q > -1, p - q < 1, \text{ as } \epsilon \to 0.$$

In particular, these restrictions are satisfied if $p = 0$, so that $d = O(1)$ and $q = 0$ or 1, so that $c = O(1)$ or $O(\epsilon)$.

From (4.10) it follows that, to first order, in the boundary layer u is a harmonic function of x and η, and v has the form

$$v = g(x) + \eta h(x). \tag{4.11}$$

It is interesting to note the duality between (4.10) and (4.11) on the one hand, and (3.10) and (3.16) of the inextensible theory on the other. In the inextensible theory u depends only on y, and v is a harmonic function of $\xi = x/c$ and y.

For small values of ϵ the inextensible theory gives the first approximation to the solution of a given suitably posed problem except in the neighbourhood of fibres which it predicts to be singular. In the neighbourhood of a fibre which in the inextensible theory is singular (the boundary layer region) the boundary layer equations (4.10) yield the first approximation to the solution.

Now consider the boundary conditions which solutions of the boundary layer equations must satisfy. Two main cases arise, according to whether the boundary layer is adjacent to a boundary surface of the body or in its interior. Suppose first that the layer is adjacent to a surface; for definiteness this is taken to be $y = 0$ (or $\eta = 0$) and the material occupies part or all of the region $y > 0$. Typical conditions on the surface are that either (i) u and v are specified on $y = 0$, or (ii) t_{yy} and t_{xy} are specified on $y = 0$.

As $\eta \to \infty$, the boundary layer solution must match the solution in the exterior region as $y \to 0$. Consider first the displacement component v. The inextensible solution yields values of v which satisfy (3.15) and also any boundary conditions on $y = 0$ for v and $\partial v/\partial y$. Suppose that as $y \to 0$ in the exterior region

$$v \to G(x), \qquad \partial v/\partial y \to H(x). \tag{4.12}$$

In the boundary layer region v is of the form (4.11). If now $g(x)$ and $h(x)$ are chosen so that

$$g(x) = G(x), \qquad h(x) = \epsilon H(x), \tag{4.13}$$

then boundary conditions on v or $\partial v/\partial y$ are satisfied on $y = 0$ and, as regards v, the boundary layer and exterior solutions coincide in the neighbourhood of $y = 0$. Thus v and its first derivatives are continuous through the boundary layer.

For the displacement component u, the exterior solution gives, from (3.10), $u \to f(0)$ as $y \to 0$. Hence for the boundary layer solution it is required that

$$u \to f(0) \quad \text{as} \quad \eta \to \infty,$$

where $f(0)$ is given by the exterior solution. On the boundary $y = 0$ typically either u is given or t_{xy} is given. Since v, and hence $\partial v/\partial x$ is known, specifying t_{xy} is equivalent to specifying $\partial u/\partial y$ on $y = 0$. Thus typical boundary conditions are that u or $\partial u/\partial \eta$ is specified on $\eta = 0$ and as $\eta \to \infty$; these of course are boundary conditions typically encountered in the solution of Laplace's equation. To complete the boundary conditions for u it is necessary to state conditions on two curves which close the boundary of the region in which the boundary layer solution is required. In a finite body these intersect $y = 0$ at finite values of x. In an infinite body conditions are specified as $x \to \pm\infty$. Various possibilities arise; no attempt is made to enumerate them, but some are illustrated by examples in the following sections.

Now suppose that the boundary layer is in the interior of a body; for definiteness let it again be in the neighbourhood of $y = 0$, with both sides of $y = 0$ now occupied by the material. The inextensible solution will yield values of v in $y > 0$ and $y < 0$; these will be such that v and $\partial v/\partial y$ are continuous. Thus

$$v \to G(x), \qquad \partial v/\partial y \to H(x) \qquad \text{as} \quad y \to 0+ \quad \text{or} \quad y \to 0-. \tag{4.14}$$

Consequently if v is again chosen in the boundary layer region to be

$$v = G(x) + \epsilon\eta H(x), \tag{4.15}$$

the continuity conditions for v and t_{yy} are satisfied. In the inextensible solution $\partial u/\partial y$ may be discontinuous across $y = 0$. Let the exterior solutions in $y > 0$; and $y < 0$ give

$$u \to f^+(0) \quad \text{as} \quad y \to 0+, \qquad u \to f^-(0) \qquad \text{as} \quad y \to 0-. \tag{4.16}$$

Then in the boundary layer u must satisfy

$$u \to f^+(0) \quad \text{as} \quad \eta \to \infty, \qquad u \to f^-(0) \qquad \text{as} \quad \eta \to -\infty. \tag{4.17}$$

As before, further conditions are required on the remaining boundaries of the boundary layer region.

Since, in the boundary layer, u is a harmonic function of the variables x and η, so also are $\partial u/\partial x$ and $\partial u/\partial \eta$ harmonic functions of x and η. It often proves convenient to formulate problems in terms of one of these derivatives of u rather than of u itself. The boundary conditions for $\partial u/\partial x$ and $\partial u/\partial \eta$ are analogous to those for u.

It is possible to formulate higher order approximations, in both the exterior and boundary layer regions. However it seems likely that when the first order approximations described above are inadequate it would be simpler to return to the exact equations rather than to use such higher approximations.

5. POINT FORCE PROBLEMS FOR THE HALF-PLANE

To illustrate the theory in some problems in which comparisons can be made with exact solutions several problems of point forces applied to the surface of a half-plane are investigated in this section.

Consider first the half-plane $x > 0$ with the fibres $y = \text{constant}$ normal to the surface $x = 0$, and suppose a point force of magnitude X acts on the surface in the positive x direction at the origin. The inextensible theory gives the trivial solution

$$u = 0, \qquad v = 0, \qquad t_{xx} = -X\delta(y), \qquad t_{yy} = 0, \qquad t_{xy} = 0. \tag{5.1}$$

Thus the fibre $y = 0$ carries the force X, and has infinite stress, and so it is the centre of a boundary layer in the interior of the material. Within this boundary layer $v = 0$, as given by the inextensible solution, and u satisfies (4.10)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2} = 0. \tag{5.2}$$

On the surface $x = 0$

$$t_{xx} = -X\delta(y) = -\frac{1}{\epsilon} X\delta(\eta), \qquad t_{xy} = 0, \tag{5.3}$$

and so, from (4.7)

$$\frac{\partial u}{\partial x} = -\epsilon \frac{X}{\mu_L} \delta(\eta), \qquad \frac{\partial u}{\partial \eta} = 0, \qquad \text{on} \quad x = 0. \tag{5.4}$$

It is also required that t_{xx} and t_{xy} tend to the values given by (5.1) as $\eta \to \pm\infty$, which with (4.7) give

$$\frac{\partial u}{\partial x} \to 0, \qquad \frac{\partial u}{\partial \eta} \to 0 \qquad \text{as} \quad \eta \to \pm\infty.$$

It is further assumed that the stress components tend to zero as $x \to \infty$, so that

$$\frac{\partial u}{\partial x} \to 0, \qquad \frac{\partial u}{\partial \eta} \to 0 \qquad \text{as} \quad x \to \infty.$$

It is most convenient to work in terms of $\partial u/\partial x$, for which the required solution is

$$\frac{\partial u}{\partial x} = -\frac{\epsilon X}{\pi \mu_L} \frac{x}{(x^2 + \eta^2)}. \tag{5.5}$$

It then readily follows that

$$\frac{\partial u}{\partial \eta} = \frac{-\epsilon X}{\pi \mu_L} \frac{\eta}{(x^2 + \eta^2)}, \qquad t_{xx} = \frac{-X}{\epsilon \pi} \frac{x}{(x^2 + \eta^2)},$$

$$t_{yy} = -\frac{\epsilon X}{d^2 \pi} \frac{x}{(x^2 + \eta^2)}, \qquad t_{xy} = -\frac{X}{\pi} \frac{\eta}{(x^2 + \eta^2)}. \tag{5.6}$$

The resultant force in the x direction transmitted across a section x = constant is

$$-\int_{-\infty}^{\infty} t_{xx}\, \mathrm{d}y = -\epsilon \int_{-\infty}^{\infty} t_{xx}\, \mathrm{d}\eta = \frac{X}{\pi} \left[\tan^{-1} \frac{\eta}{x}\right]_{-\infty}^{\infty} = X,$$

so that the boundary layer carries the same force as the singular fibre $y = 0$ in the inextensible theory.

For this problem there exists a well-known solution of the exact equations (2.9) of anisotropic plane elasticity. This is most conveniently taken in the form used by Everstine and Pipkin[5], which is

$$t_{rr} = -\frac{X}{\pi r} \frac{\epsilon_t (1 + \epsilon_c \epsilon_t) \cos\theta}{(\cos^2\theta + \epsilon_c^{\,2} \sin^2\theta)(\sin^2\theta + \epsilon_t^{\,2} \cos^2\theta)}, \qquad t_{\theta\theta} = t_{r\theta} = 0, \tag{5.7}$$

where $x = r\cos\theta$, $y = r\sin\theta$, t_{rr}, $t_{\theta\theta}$, $t_{r\theta}$ are stress components referred to (r, θ) coordinates, and in the present notation the parameters ϵ_t and ϵ_c introduced by Everstine and Pipkin are such that $\epsilon_t^{\,-2}$ and $\epsilon_c^{\,2}$ are the larger and smaller roots λ of the equation

$$\frac{\lambda^2}{c^2} + \left(\frac{2}{d^2} + \frac{1}{d^4} - \frac{1}{\epsilon^2 c^2}\right)\lambda + \frac{1}{\epsilon^2} = 0. \tag{5.8}$$

Everstine and Pipkin showed that the exact solution (5.7) tends to the inextensible theory solution (5.1) in the limit $\epsilon_t \to 0$. It is also evident, on setting $\eta = y/\epsilon$, that the boundary layer solution (5.6) tends to (5.1) as $\epsilon \to 0$. To complete the comparison it is necessary to compare the boundary layer solution with the exact solution. The difference between the two is greatest on the axis of symmetry $y = 0$ (or $\theta = 0$) where the boundary layer solution (5.6) gives

$$t_{xx} = -\frac{X}{\epsilon \pi x} \tag{5.9}$$

and the exact solution (5.7) gives

$$t_{rr} = t_{xx} = -\frac{(1 + \epsilon_c \epsilon_t) X}{\epsilon_t \pi x}. \tag{5.10}$$

Thus the boundary layer solution replaces the exact coefficient $(1 + \epsilon_c \epsilon_t)/\epsilon_t$ by $1/\epsilon$. From (5.8) it can readily be shown that

$$\epsilon_t = \epsilon + O(\epsilon^3), \qquad \epsilon_c = c + O(\epsilon^2). \tag{5.11}$$

and so the approximation amounts essentially to neglecting ϵ in comparison to unity. Using the values of the elastic constants for a carbon fibre-epoxy composite measured by Markham[11] and quoted in Section 4, Bishop[9] obtained, for plane stress $\epsilon_t = 0{\cdot}157$ and $\epsilon_c = 0{\cdot}883$. With the same elastic constants $\epsilon = 0{\cdot}155$ and $c = 0{\cdot}874$. Thus the error in (5.9) in this case arises almost entirely from omitting the term $\epsilon_c \epsilon_t$ in the numerator of (5.10), and in magnitude is about 14 per cent.

From (5.5) and (5.6) the displacement component u is, except for a constant

$$u = \frac{-\epsilon X}{2\pi\mu_L} \log(x^2 + \eta^2). \tag{5.12}$$

The logarithmic singularity in u at the origin is present also in the exact solution, so its presence in the boundary layer solution does not give rise to concern.

A point force acting on the boundary $x = 0$ in the y-direction does not lead to singular fibres in the inextensible theory, and so no boundary layers are to be expected in this case.

Now consider point forces applied to the half-plane $y > 0$ with the fibres again parallel to the x-axis. Suppose first that a point force X acts at the origin in the positive x-direction. The inextensible theory gives the solution

$$\begin{gathered} u = 0, \qquad v = 0, \qquad t_{xy} = 0, \qquad t_{yy} = 0, \\ t_{xx} = -\tfrac{1}{2} X\delta(y) \quad (x > 0), \qquad t_{xx} = \tfrac{1}{2} X\delta(y) \quad (x < 0). \end{gathered} \tag{5.13}$$

It can be verified that in this case also the solution (5.6) of the boundary layer equations satisfies all the conditions of the problem. Also, (5.7) is again the exact solution. The discussion of these solutions therefore applies also to this problem.

Of more interest is the case in which a point force Y acts at the origin in the positive y-direction. The inextensible solution was given by England, Ferrier and Thomas[3] and is

$$\begin{aligned} &u = 0, \qquad v = \frac{-cY}{2\pi\mu_L} \log(x^2 + c^2 y^2), \\ &t_{xx} = \frac{cY}{\pi}\left[\delta(y) \log x + \frac{c^2 y}{x^2 + c^2 y^2}\right], \\ &t_{yy} = -\frac{cY}{\pi}\frac{y}{(x^2 + c^2 y^2)}, \qquad t_{xy} = -\frac{cY}{\pi}\frac{x}{(x^2 + c^2 y^2)} H(y + 0), \end{aligned} \tag{5.14}$$

where $H(y + 0)$ is the Heaviside step function, equal to one when $y > 0$ and to zero when $y \leqslant 0$. The edge fibre $y = 0$ is a singular fibre.

The expression for v holds for all $y > 0$, including the boundary layer. Since $t_{xy} = 0$ on $y = 0$, we require

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{on} \quad y = 0.$$

From (5.14), since $y = \epsilon\eta$, this gives

$$\frac{\partial u}{\partial \eta} = \frac{\epsilon c Y}{\pi \mu_L x} \quad \text{on} \quad \eta = 0. \tag{5.15}$$

This is one boundary condition for the boundary layer. In order to match the boundary layer solution with the inextensible solution (5.14) it is also necessary that

$$\frac{\partial u}{\partial \eta} \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{5.16}$$

The solution of (5.2) subject to (5.15) and (5.16) is

$$\frac{\partial u}{\partial \eta} = \frac{\epsilon c Y}{\pi \mu_L} \frac{x}{(x^2 + \eta^2)}. \tag{5.17}$$

Consequently the boundary layer solution is

$$\begin{gathered} u = \frac{\epsilon c Y}{\pi \mu_L} \tan^{-1} \frac{\eta}{x}, \\ t_{xx} = -\frac{cY}{\pi\epsilon} \frac{\eta}{(x^2 + \eta^2)}, \qquad t_{xy} = \frac{cY}{\pi} \left\{ \frac{x}{x^2 + \eta^2} - \frac{x}{x^2 + c^2 y^2} \right\}, \end{gathered} \tag{5.18}$$

with v and t_{yy} given by (5.14).

The exact solution to this problem is obtained from (5.7) by interchanging ϵ_c and ϵ_t, replacing X by Y and setting $x = r \sin\theta$, $y = r\cos\theta$. This gives in the neighbourhood of the singular fibre $y = 0$

$$t_{xx} = -\frac{Y}{\pi} \frac{\epsilon_c(1 + \epsilon_c \epsilon_t) y}{y^2 + \epsilon_t^2 x^2},$$

which, after making use of (5.11), agrees with the expression for t_{xx} given in (5.18) to first order in ϵ. Similarly, on the axis of symmetry $x = 0$, the exact solution gives

$$t_{yy} = -\frac{Y}{\pi y} \frac{(1 + \epsilon_c \epsilon_t)}{\epsilon_c}$$

which agrees with the value given by (5.14) to first order in ϵ.

Various other point force problems can be dealt with in a similar manner, such as those of point forces acting in the interior of the whole plane, the half-plane $x > 0$, the half-plane $y > 0$, or at the surface of or in the interior of an infinite strip. Many other problems can then be solved by superpositions of these basic point force solutions, although this may not necessarily be the easiest way to proceed in a given problem.

6. A CRACK PARALLEL TO THE FIBRES IN SHEAR

Consider an infinite plane region containing fibres parallel to the x-axis. A crack extends along the x-axis $y = 0$ from $x = -a$ to $x = a$, and its surface is free from traction. As $x^2 + y^2 \to \infty$ the state of stress and deformation tends to one of simple shear

$$u = yS/\mu_L, \qquad v = 0, \qquad t_{xy} = S, \qquad t_{xx} = t_{yy} = 0.$$

If a uniform displacement $u = -yS/\mu_L$, with the corresponding stress $t_{xy} = -S$, is superposed on this deformation, then the boundary conditions become

$$\begin{aligned} &t_{xy} = -S, \quad t_{yy} = 0, \quad y = 0, \quad -a < x < a. \\ &u, v, t_{xx}, t_{yy}, t_{xy} \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \end{aligned} \tag{6.1}$$

It is clearly sufficient to solve this modified problem. The original problem is then solved by superimposing a uniform shear stress and strain on the solution of the modified problem.

The problem is symmetrical about $y = 0$, so it is sufficient to consider the region $y \geqslant 0$. The solution of the equations of the inextensible theory which satisfies (6.1) is one of zero stress and displacement everywhere except on the fibre $y = 0$, $-a < x < a$. This fibre is singular, carrying a tensile force $-Sx$ (with corresponding tensile stress $-Sx\,\delta(y)$) and shear stress $-S$. Consequently there is a boundary layer in the vicinity of this fibre segment.

The solution $v = 0$ extends through the boundary layer. Hence the condition $t_{xy} = -S$ on $y = 0$, $|x| < a$, reduces to

$$\mu_L \frac{\partial u}{\partial y} = -S, \quad \text{or} \quad \frac{\partial u}{\partial \eta} = -\epsilon \frac{S}{\mu_L}, \quad y = 0, \quad |x| < a. \tag{6.2}$$

On the remainder of the line $y = 0$, by symmetry

$$u = 0, \quad y = 0, \quad |x| > a. \tag{6.3}$$

It is also required that

$$u \to 0, \quad x^2 + y^2 \to \infty. \tag{6.4}$$

Thus the boundary layer problem reduces to solving Laplace's equation (4.10) subject to (6.2), (6.3) and (6.4). The solution is known, and can be expressed in various forms, one of which is

$$u = -\epsilon \frac{S}{\mu_L} \{\rho \sin \phi - (\rho_1 \rho_2)^{1/2} \sin \tfrac{1}{2}(\phi_1 - \phi_2)\}, \tag{6.5}$$

where

$$x + i\eta = \rho^{i\phi}, \quad x - a + i\eta = \rho_1 e^{i\phi_1}, \quad x + a + i\eta = \rho_2 e^{i\phi_2}. \tag{6.6}$$

The stress components can be obtained by substituting (6.5) into (4.7). The component of greatest interest is t_{xx} on the surface $y = 0$, $|x| < a$. From (6.5)

$$u = \epsilon \frac{S}{\mu_L} (a^2 - x^2)^{1/2}, \quad y = 0, \quad |x| < a. \tag{6.7}$$

Hence, from (4.7)

$$t_{xx} = -\frac{S}{\epsilon} \frac{x}{(a^2 - x^2)^{1/2}}, \quad y = 0, \quad |x| < a. \tag{6.8}$$

At $x = \pm a$, t_{xx} has singularities of the form often encountered in elastic crack problems.

In the neighbourhood of the crack tip $(a, 0)$, $\rho \approx a$, $\phi \approx 0$, $\rho_2 \approx 2a$, $\phi_2 \approx 0$, and

$$u \approx \frac{\epsilon S}{\mu_L} \{(2a\rho_1)^{1/2} \sin \tfrac{1}{2}\phi_1 - \eta\}. \tag{6.9}$$

The corresponding stress components in the vicinity of the crack tip are

$$t_{xx} = -\frac{S}{\epsilon}\left(\frac{a}{2\rho_1}\right)^{1/2} \sin \tfrac{1}{2}\phi_1, \qquad t_{yy} = 0,$$
$$t_{xy} = S\left\{\left(\frac{a}{2\rho_1}\right)^{1/2} \cos \tfrac{1}{2}\phi_1 - 1\right\}. \tag{6.10}$$

The solution to the original problem of the crack subject to uniform shear stress at infinity is obtained by superimposing on this solution a uniform stress $t_{xy} = S$ and a displacement $u = Sy/\mu_L$.

7. CRACK NORMAL TO THE FIBRES UNDER INTERNAL PRESSURE

Suppose an infinite plane region, with fibres again parallel to the x-axis, contains a crack lying along the y-axis from $y = -a$ to $y = a$. No shear traction is applied to the crack surface and, in the first instance it is assumed that the surface is subjected to a prescribed normal displacement, so that the boundary conditions are

$$t_{xy} = 0, \qquad u = f(y) \qquad \text{on} \quad x = 0, \qquad |y| < a. \tag{7.1}$$

In plane strain of an ideal fibre-reinforced material, problems of this kind are discussed by England and Rogers[10]. For continuity of displacement, it is necessary that $f(a) = f(-a) = 0$. For simplicity, it is assumed that f is an even function of y. Since the configuration is symmetrical, it is sufficient to consider the region $x \geqslant 0$.

The inextensible solution gives first

$$\begin{aligned} u &= f(y), \qquad |y| < a,\\ u &= 0 \qquad\quad\ \ |y| > a, \end{aligned} \tag{7.2}$$

throughout the region. In the inextensible solution v must now be chosen to satisfy the condition $t_{xy} = 0$ on $x = 0$, for $-\infty < y < \infty$. With (7.2) this means

$$\begin{aligned} \frac{\partial v}{\partial x} &= -f'(y), \qquad |y| < a.\\ \frac{\partial v}{\partial x} &= 0, \qquad\qquad |y| > a. \end{aligned} \tag{7.3}$$

The solution of (3.15) which satisfies these, with $\partial v/\partial x \to 0$ as $x^2 + y^2 \to \infty$, is

$$\frac{\partial v}{\partial x} = -\frac{1}{\pi}\int_{-a}^{a} \frac{\xi f'(y')\mathrm{d}y'}{\xi^2 + (y - y')^2}. \tag{7.4}$$

For example, if the crack has a parabolic profile

$$f(y) = U_0\left(1 - \frac{y^2}{a^2}\right), \qquad f'(y) = -2U_0 y/a^2, \tag{7.5}$$

then (7.4) gives

$$\frac{\partial v}{\partial x} = \frac{2U_0}{\pi a^2}\left[-y\left\{\tan^{-1}\frac{y - a}{\xi} - \tan^{-1}\frac{y + a}{\xi}\right\} + \tfrac{1}{2}\xi \log\left\{\frac{\xi^2 + (y - a)^2}{\xi^2 + (y + a)^2}\right\}\right]. \tag{7.6}$$

However, an explicit knowledge of v is not necessary in order to find the boundary layer solution,

Still within the inextensible solution, the stress components are given by

$$\begin{aligned} t_{yy} &= N\partial v/\partial y, \\ t_{xy} &= \mu_L \partial v/\partial x, \qquad |y| > a, \\ t_{xy} &= \mu_L(f'(y) + \partial v/\partial x), \qquad |y| < a, \\ t_{xx} &= F(y) - \int (\partial t_{xy}/\partial y)\, \mathrm{d}x. \end{aligned} \tag{7.7}$$

The arbitrary function $F(y)$ can be chosen to satisfy a boundary condition on t_{xx}. We observe, however, that if $f''(y) \neq 0$ and $F(y)$ is bounded, then $|t_{xx}| \to \infty$ as $x \to \infty$ in $|y| < a$, so that, as noted in[3], there are difficulties associated with applying the inextensible theory to infinite regions. We return to these later. Since, when $f'(a) \neq 0$, t_{xy} is discontinuous on $y = \pm a$, the fibres $y = \pm a$ are singular, and carry a force

$$T = T_0 + \mu_L x f'(y). \tag{7.8}$$

Observe again that $|T| \to \infty$ as $x \to \infty$.

Now consider the boundary layer solution in the vicinity of $y = a$. From (7.7) it follows that in the boundary layer there is required a solution of (4.10) in $x \geqslant 0$ which satisfies

$$\begin{aligned} \frac{\partial u}{\partial y} &\to f'(a), \qquad \text{as} \quad \eta_1 \to -\infty, \\ \frac{\partial u}{\partial y} &\to 0, \qquad \text{as} \quad \eta_1 \to \infty, \end{aligned} \tag{7.9}$$

where for convenience we use $\eta_1 = (y - a)/\epsilon = \eta - a/\epsilon$ as an independent variable. In addition, the symmetry of the configuration requires that

$$u = 0, \qquad \text{when} \quad x = 0, \qquad \eta_1 > 0, \tag{7.10}$$

and for u to have its prescribed value on $x = 0$ for $\eta_1 < 0$ in the neighbourhood of $y = a$, it is necessary that

$$u = \epsilon \eta_1 f'(a), \qquad \text{when} \quad x = 0, \qquad \eta_1 < 0. \tag{7.11}$$

The required solution is

$$\frac{\partial u}{\partial \eta_1} = \epsilon \left(\frac{1}{2} - \frac{1}{\pi} \tan^{-1} \frac{\eta_1}{x}\right) f'(a) \tag{7.12}$$

or

$$u = \epsilon \left(\frac{1}{2}\eta_1 - \frac{1}{\pi}\eta_1 \tan^{-1}\frac{\eta_1}{x} + \frac{1}{2\pi} x \log(x^2 + \eta_1^2) + Cx\right) f'(a), \tag{7.13}$$

where C is a constant which can be chosen to satisfy an additional boundary condition. The stress components are determined from (7.4) and (7.13), but details are omitted for reasons given below.

There are two difficulties associated with this solution. The first is that the boundary layers increase in width with increasing distance from the crack tips, and the two boundary layers corresponding to the singular fibres $y = a$ and $y = -a$ begin to interact at distances of order a/ϵ from the crack tips. Hence the solution becomes invalid at such distances.

The second problem is associated with the assumption that the elastic region is infinite. Suppose for definiteness that the crack is opened by internal pressure, with zero traction on a boundary at large distance, of order l, from the crack tip; this can be converted to the problem of a crack opened by traction applied to the outer boundary in the usual way. Then from (7.7) the pressure required to open the crack is of order $-\mu_L l f''(y)$, and tends to infinity as $l \to \infty$. On the one hand it is necessary for l/a to be large in order that the solution for v and the boundary layer solution for u shall be valid; on the other hand, if l is large, then t_{xx} is large in magnitude for $|y| < a$ and small values of x. But if $|t_{xx}|$ is sufficiently large then $\partial u/\partial x$ is not small even if ϵ is small, and the assumption on which the inextensible theory is based becomes invalid.

It is possible to resolve these difficulties and produce a uniformly valid approximate solution in the following way. Near the crack $|t_{xx}|$ is large and $\partial^2 u/\partial x^2$ is not necessarily negligible compared to $\partial^2 u/\partial \eta^2$. On the other hand $\partial^2 v/\partial x\,\partial \eta$ remains small. Hence in the exact relation (4.8), the term in $\partial^2 v/\partial x\,\partial \eta$ may be neglected but the term in $\partial^2 u/\partial x^2$ should be retained. This gives the boundary layer equation (4.10). At large distances from the crack, as $t_{xx} \to 0$, the term $\partial^2 u/\partial x^2$ is small compared to $\partial^2 u/\partial \eta^2$. However, there is no loss in retaining this term, even though it has little effect on the solution. Hence, in order to have the same equation for u throughout the plane, the approximate equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2} = 0 \tag{7.14}$$

is adopted everywhere.

With this formulation, the boundary conditions for u (with $y = \epsilon\eta$) are

$$\begin{aligned} u &= 0, & x &= 0, & |y| &> a, & |\eta| &> a/\epsilon, \\ u &= f(\epsilon\eta), & x &= 0, & |y| &< a, & |\eta| &< a/\epsilon, \\ \partial u/\partial x &\to 0, & \partial u/\partial \eta &\to 0, & x^2 + \eta^2 &\to \infty. \end{aligned} \tag{7.15}$$

The solution of (7.14) which satisfies these is

$$u = \frac{1}{\pi}\int_{-a/\epsilon}^{a/\epsilon} \frac{x f(\epsilon\eta')\,d\eta'}{x^2 + (\eta - \eta')^2}, \tag{7.16}$$

so that if, for example, the crack has the parabolic profile (7.5)

$$u = \frac{U_0}{\pi}\left[\left\{1 - \frac{\epsilon^2(\eta^2 - x^2)}{a^2}\right\}\left\{\tan^{-1}\frac{\eta + \epsilon^{-1}a}{x} - \tan^{-1}\frac{\eta - \epsilon^{-1}a}{x}\right\} - \frac{\epsilon^2 \eta x}{a^2}\log\left\{\frac{x^2 + (\eta - \epsilon^{-1}a)^2}{x^2 + (\eta + \epsilon^{-1}a)^2}\right\} - \frac{2\epsilon x}{a^2}\right].$$

By expanding $f(\epsilon\eta)$ about $y = \epsilon\eta = a$, it can be shown after a little manipulation that if $\eta \ll a/\epsilon$, $x \ll a/\epsilon$, then to first order in ϵ the expression for $\partial u/\partial \eta$ given by (7.16) reduces to (7.12). Thus the two methods of approximation are equivalent within the boundary layer region.

When the problem is formulated in this way it becomes straightforward to deal with the more realistic problem in which the pressure, rather than the normal displacement, is specified on the surface of the crack. Suppose a pressure $p(y)$ is applied to the surface, so that $t_{xx} = -p(y)$ on $x = 0$, $|y| < a$. It follows from (4.7) that

$$\frac{1}{\epsilon^2}\frac{\partial u}{\partial x} + \frac{1}{d^2}\frac{\partial v}{\partial y} = -\frac{p(y)}{\mu_L}, \qquad x = 0, \qquad |y| < a. \tag{7.17}$$

It is consistent with the boundary layer approximation to neglect $\epsilon^2\, \partial v/\partial y$ compared to $d^2\, \partial u/\partial x$ (the validity of this can be verified *a posteriori*), and so the boundary conditions for u become

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{\epsilon^2}{\mu_L} p(y), \qquad x = 0, \qquad |y| < a, \\ u &= 0, \qquad x = 0, \qquad |y| > a. \end{aligned} \tag{7.18}$$

We consider the case in which the crack is opened by a uniform internal pressure, and $p(y) = p_0$, where p_0 is constant. Then the solution of (7.14) subject to (7.18), with $\partial u/\partial x$ and $\partial u/\partial \eta \to 0$ as $x^2 + \eta^2 \to \infty$ is

$$u = -\frac{\epsilon^2 p_0}{\mu_L}\{x - (r_1 r_2)^{1/2} \cos \tfrac{1}{2}(\theta_1 + \theta_2)\}, \tag{7.19}$$

where

$$\begin{aligned} x + i(\eta - \epsilon^{-1} a) &= r_1 e^{i\theta_1}, \\ x + i(\eta + \epsilon^{-1} a) &= r_2 e^{i\theta_2}. \end{aligned} \tag{7.20}$$

On $x = 0$, $|y| < a$, (7.19) gives

$$u = \frac{\epsilon^2 p_0}{\mu_L}(\epsilon^{-2} a^2 - \eta^2)^{1/2} = \frac{\epsilon p_0}{\mu_L}(a^2 - y^2)^{1/2} \tag{7.21}$$

so that the crack has the characteristic elliptical shape. From (7.21)

$$\frac{\partial u}{\partial y} = -\frac{\epsilon p_0 y}{\mu_L (a^2 - y^2)^{1/2}}, \qquad x = 0, \qquad |y| < a, \tag{7.22}$$

and so, from (7.3), the boundary conditions for v are

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\epsilon p_0 y}{\mu_L (a^2 - y^2)^{1/2}}, \qquad x = 0, \qquad |y| < a, \\ \frac{\partial v}{\partial x} &= 0, \qquad x = 0, \qquad |y| > a. \end{aligned} \tag{7.23}$$

The solution of (3.16) which satisfies (7.23) together with the symmetry condition $v = 0$ on $y = 0$ and with $\partial v/\partial y \to 0$ as $x^2 + y^2 \to \infty$ is

$$v = = \frac{\epsilon c p_0}{\mu_L}\{y - (s_1 s_2)^{1/2} \sin \tfrac{1}{2}(\psi_1 + \psi_2)\}, \tag{7.24}$$

where

$$\begin{aligned} \xi + i(y - a) &= s_1 e^{i\psi_1}, \\ \xi + i(y + a) &= s_2 e^{i\psi_2}. \end{aligned} \tag{7.25}$$

It can now be verified that $\partial v/\partial y$ is of order ϵ times $\epsilon^{-2}\, \partial u/\partial x$, so it is justified to neglect the term in $\partial v/\partial y$ in (7.17).

From (7.19) and (7.24) there follows

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -\frac{\epsilon^2 p_0}{\mu_L}\left\{1 - \frac{1}{2}\frac{r_1 + r_2}{(r_1 r_2)^{1/2}} \cos \tfrac{1}{2}(\theta_1 - \theta_2)\right\}, \\
\frac{\partial u}{\partial y} &= -\frac{\epsilon p_0}{2\mu_L}\frac{r_1 - r_2}{(r_1 r_2)^{1/2}} \sin \tfrac{1}{2}(\theta_1 - \theta_2), \\
\frac{\partial v}{\partial x} &= \frac{\epsilon p_0}{2\mu_L}\frac{s_1 - s_2}{(s_1 s_2)^{1/2}} \sin \tfrac{1}{2}(\psi_1 - \psi_2), \\
\frac{\partial v}{\partial y} &= -\frac{\epsilon c p_0}{\mu_L}\left\{1 - \frac{1}{2}\frac{s_1 + s_2}{(s_1 s_2)^{1/2}} \cos \tfrac{1}{2}(\psi_1 - \psi_2)\right\},
\end{aligned} \tag{7.26}$$

which gives the stress components, to leading order in ϵ

$$\begin{aligned}
t_{xx} &= -p_0\left\{1 - \frac{1}{2}\frac{r_1 + r_2}{(r_1 r_2)^{1/2}} \cos \tfrac{1}{2}(\theta_1 - \theta_2)\right\}, \\
t_{yy} &= -\frac{\epsilon}{c}\, p_0\left\{1 - \frac{1}{2}\frac{s_1 + s_2}{(s_1 s_2)^{1/2}} \cos \tfrac{1}{2}(\psi_1 - \psi_2)\right\}, \\
t_{xy} &= -\frac{\epsilon p_0}{2}\left\{\frac{r_1 - r_2}{(r_1 r_2)^{1/2}} \sin \tfrac{1}{2}(\theta_1 - \theta_2) - \frac{s_1 - s_2}{(s_1 s_2)^{1/2}} \sin \tfrac{1}{2}(\psi_1 - \psi_2)\right\}.
\end{aligned} \tag{7.27}$$

The quantities of most interest in this problem are probably the stress components in the immediate vicinity of the tips of the crack. Let r, θ be polar co-ordinates with origin at the crack tip $x = 0$, $y = a$, so that

$$x + i(y - a) = re^{i\theta}. \tag{7.28}$$

Then from (7.20) and (7.25)

$$\begin{aligned}
{r_1}^2 &= r^2(\cos^2\theta + \epsilon^{-2}\sin^2\theta), \qquad \tan\theta_1 = \epsilon^{-1}\tan\theta, \\
{s_1}^2 &= r^2(c^{-2}\cos^2\theta + \sin^2\theta), \qquad \tan\psi_1 = c\tan\theta,
\end{aligned} \tag{7.29}$$

and, for $r/a \ll 1$,

$$r_2 \approx 2\epsilon^{-1}a, \qquad \theta_2 \approx \tfrac{1}{2}\pi, \qquad s_2 \approx 2a, \qquad \psi_2 \approx \tfrac{1}{2}\pi. \tag{7.30}$$

Hence for $r_1/a \ll 1$, (7.27) are, approximately

$$\begin{aligned}
t_{xx} &= -p_0 + p_0\left(\frac{a}{2r}\right)^{1/2}\frac{\cos(\frac{1}{2}\theta_1 - \frac{1}{4}\pi)}{(\epsilon^2\cos^2\theta + \sin^2\theta)^{1/4}}, \\
t_{yy} &= -\frac{\epsilon}{c}\, p_0 + \frac{\epsilon}{c}\, p_0\left(\frac{a}{2r}\right)^{1/2}\frac{\cos(\frac{1}{2}\psi_1 - \frac{1}{4}\pi)}{(c^{-2}\cos^2\theta + \sin^2\theta)^{1/4}}, \\
t_{xy} &= \epsilon p_0\left(\frac{a}{2r}\right)^{1/2}\left\{\frac{\sin(\frac{1}{2}\theta_1 - \frac{1}{4}\pi)}{(\epsilon^2\cos^2\theta + \sin^2\theta)^{1/4}} - \frac{\sin(\frac{1}{2}\psi_1 - \frac{1}{4}\pi)}{(c^{-2}\cos^2\theta + \sin^2\theta)^{1/4}}\right\}.
\end{aligned} \tag{7.31}$$

The variation of the stress components t_{xx} and t_{xy} (t_{yy} is of less interest) with θ is shown in Figs 1 and 2 for the material constants given by Markham[11] for a typical carbon fibre-

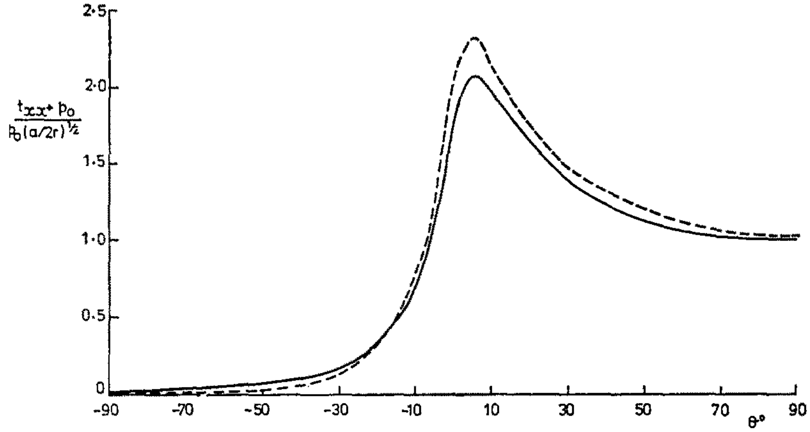

Fig. 1. Shows the variation of t_{xx} with the polar angle θ from the crack tip. The crack lies on $x = 0$, $|y| < a$; $x + i(y - a) = r \exp(i\theta)$ and p_0 is the pressure applied to the crack surface. The broken line shows the exact solution and the solid line the boundary layer solution. Elastic constants for a carbon-fibre reinforced epoxy resin.

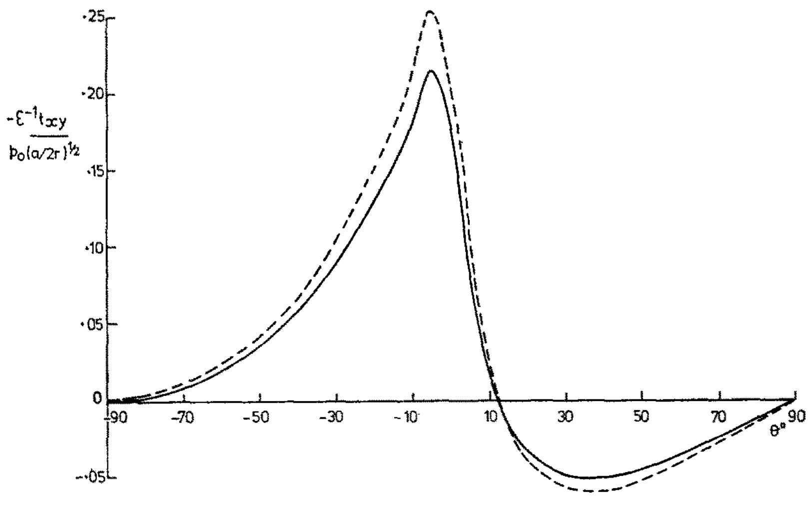

Fig. 2. Shows the variation of t_{xy} with θ. Notation as for Fig. 1.

epoxy composite, which give

$$\epsilon = 0{\cdot}155, \qquad c = 0{\cdot}874. \tag{7.32}$$

The exact solution to this problem is also known (Bishop[9] gives extensive numerical results and references to original papers). In the neighbourhood of the crack tip, $r/a \ll 1$, the results analogous to (7.31) given by the exact solution are

$$
\begin{aligned}
t_{xx} &= -p_0 + \frac{p_0}{1-\epsilon_c\epsilon_t}\left(\frac{a}{2r}\right)^{1/2}\left\{\frac{\cos(\frac{1}{2}\gamma - \frac{1}{4}\pi)}{(\epsilon_t^2\cos^2\theta + \sin^2\theta)^{1/4}} - \epsilon_c\epsilon_t\frac{\cos(\frac{1}{2}\delta - \frac{1}{4}\pi)}{(\epsilon_c^{-2}\cos^2\theta + \sin^2\theta)^{1/4}}\right\},\\
t_{yy} &= -\frac{\epsilon_t}{\epsilon_c}p_0 - \frac{\epsilon_t p_0}{1-\epsilon_c\epsilon_t}\left(\frac{a}{2r}\right)^{1/2}\left\{\frac{\epsilon_t\cos(\frac{1}{2}\gamma - \frac{1}{4}\pi)}{(\epsilon_t^2\cos^2\theta + \sin^2\theta)^{1/4}} - \frac{\cos(\frac{1}{2}\delta - \frac{1}{4}\pi)}{\epsilon_c(\epsilon_c^{-2}\cos^2\theta + \sin^2\theta)^{1/4}}\right\},\\
t_{xy} &= \frac{\epsilon_t p_0}{1-\epsilon_c\epsilon_t}\left(\frac{a}{2r}\right)^{1/2}\left\{\frac{\sin(\frac{1}{2}\gamma - \frac{1}{4}\pi)}{(\epsilon_t^2\cos^2\theta + \sin^2\theta)^{1/4}} - \frac{\sin(\frac{1}{2}\delta - \frac{1}{4}\pi)}{(\epsilon_c^{-2}\cos^2\theta + \sin^2\theta)^{1/4}}\right\},
\end{aligned}
\tag{7.33}
$$

where

$$\tan\gamma = \epsilon_t^{-1}\tan\theta, \qquad \tan\delta = \epsilon_c\tan\theta. \tag{7.34}$$

By using (5.11) it can be verified that (7.33) reduce to (7.31) to leading order in ϵ as $\epsilon \to 0$. The values of t_{xx} and t_{xy} given by (7.33) are also shown in Figs. 1 and 2 for the values

$$\epsilon_t = 0{\cdot}157, \qquad \epsilon_c = 0{\cdot}883,$$

which follow from the elastic constants given by Bishop[9] and Markham[11].

On comparing (7.31) and (7.33) it can be seen that the boundary layer approximation involves replacing ϵ_t and ϵ_c by ϵ and c respectively, and omitting certain terms. For the constants used here, the substitution of ϵ and c for ϵ_t and ϵ_c has very little effect (at most 2 per cent) on the numerical values of the stress components, and the discrepancy between the exact and approximate solutions results from the omission of terms of order ϵ compared to one. For t_{xy}, the approximation is essentially to replace the multiplier $\epsilon_t/(1-\epsilon_c\epsilon_t)$ in (7.33) by ϵ in (7.31); for the constants used this means the boundary layer theory underestimates the magnitude of t_{xy} by about 14 per cent. For t_{xx} there are two effects; in the approximation the last term (with the factor $\epsilon_c\epsilon_t$) in (7.33) is omitted, and the factor $(1-\epsilon_c\epsilon_t)^{-1}$ in (7.33) is replaced by one in (7.31). These two errors tend to cancel, leaving an overall error of less than 14 per cent. These errors would of course be less if a smaller value of ϵ had been chosen.

This problem may be regarded as a stringent test of the boundary layer theory, for it involves a complicated singularity at the crack tip in which t_{xy} has an infinite discontinuity. In these circumstances the agreement between exact and boundary layer solutions, for a value of ϵ which is not very small, is considered to be very satisfactory.

8. DISCUSSION

As pointed out by Everstine and Pipkin[6] the boundary layer analysis offers a theory intermediate in difficulty between the simple inextensible theory and the exact anisotropic theory. The examples discussed in this paper are ones for which solutions to the full anisotropic problem are available; they were deliberately so chosen in order to enable comparisons with the exact theory to be made.

For problems in which an exact solution of the anisotropic theory is not available or readily obtained, the boundary layer approach seems to offer a number of advantages (always provided, of course, that the parameter ϵ is sufficiently small). It requires only solutions of Laplace's equation, and it appears that in many cases familiar standard solutions of this equation will suffice. It is often possible to analyse the stress and deformation in the boundary layer without requiring a complete solution elsewhere. For example, to find the boundary layer displacement u, given by (7.13), in the problem of a crack opened by internal pressure, it is only necessary to know the slope $f'(a)$ of the crack surface near the tip. Since the boundary layers are usually the regions of greatest stress, it is often sufficient to be able to analyse these regions. For problems which require numerical solutions, the solution of Laplace's equation is a simpler proposition than that of the generalized biharmonic equation, and the coordinate scaling introduced in the boundary layer analysis avoids the appearance of large stress and displacement gradients which might cause difficulty and loss of accuracy in numerical solution of the exact equations.

Of course the inextensible theory is even easier to apply and this, together with the order of magnitude estimates given by Everstine and Pipkin[5], gives adequate information for many applications.

REFERENCES

1. A. C. Pipkin and T. G. Rogers, Plane deformations of incompressible fibre-reinforced materials, *J. appl. Mech.* **38**, 364 (1971).
2. A. J. M. Spencer, *Deformations of Fibre-reinforced Materials*. Clarendon Press, Oxford (1972).
3. A. H. England, J. E. Ferrier and J. N. Thomas, Plane strain and generalized plane stress problems for fibre-reinforced materials, *J. Mech. Phys. Solids* **21**, 279 (1973.)
4. L. W. Morland, A plane theory of inextensible transversely isotropic elastic composites. *Int. J. Solids Struct.* **9**, 1501 (1973).
5. G. C. Everstine and A. C. Pipkin, Stress channeling in transversely isotropic elastic composites, *Z. angew. Math. Phys.* **22**, 825 (1971).
6. G. C. Everstine and A. C. Pipkin, Boundary layers in fiber-reinforced materials, *J. appl. Mech.* **40**, 518 (1973).
7. S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Body*. Holden-Day, San Francisco (1963).
8. A. H. England, The stress boundary value problem for plane strain deformations of an ideal fibre-reinforced material, *J. Inst. Math. and its appl.* **9**, 310 (1972).
9. S. M. Bishop, Stresses near an elliptical hole in an orthotropic sheet, R.A.E. *Tech. Rep.* TR72026 (1972).
10. A. H. England and T. G. Rogers, Plane crack problems for ideal fibre-reinforced materials. *Q. Jl. Mech. appl. Math.* **26**, 303 (1973).
11. M. F. Markham, Measurement of the elastic constants of fibre composites by ultrasonics. *Composites* **1**, 145 (1970).

Абстракт — Определяется для задачи плоской деформации метод пограничного слоя, предложенный Зверстайном и Пиникином для анализа высоко анизотропных материалов, таких как материалы усиленные волокнами. Далеел обобщается этот метод, с заключения, также, плоского напряженного состояния. Метод применяется к задачам сосредоточенных усилий, действующих на полуплоскостях, и к задачам двух трещин. Сравниваются решения пограничного слоя с хорошо известными решениями в рамках теории упругости анизотропного тела. Находится, что теория пограничного слоя дает надлежащие результаты для упругих постоянных, типа угольной смолы, усиленной волокнами.